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# Convergence of the Viscosity Solutions for the System of Nonlinear Elasticity*

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In this paper some special entropy pairs of Lax type are constructed for the system of nonlinear elasticity, in which the progressive terms are functions of a single variable. The necessity estimates for the major terms are obtained by using the theory of singular perturbation of the ordinary differential equations. The special entropy pairs are used to prove that the family of Young measures uniquely determined by the viscosity solution sequence is a family of Dirac measures. Hence, a convergence theorem on the viscosity solutions is established by applying the method of compensated compactness. © 1997 Academic Press

## 1. INTRODUCTION

In this paper, we consider the existence of the global weak solutions for the system of nonlinear elasticity with linearly degenerate points, i.e., the system describing the balance of mass and momentum of the medium

$$v_t - u_x = 0, \qquad u_t - \sigma_x(v) = 0, \tag{1.1}$$

with initial data

$$(v(x,0), u(x,0)) = (v_0(x), u_0(x)), \tag{1.2}$$

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where $v$ is the strain, $u$ the velocity, and $\sigma(v)$ the stress. Our assumptions on $\sigma(v)$ are as follows:

($A_1$) $\sigma(v)$ is sufficiently smooth and $\sigma'(v) > 0$, for any $v \in \mathbf{R}$,

($A_2$) there exists a finite or infinite set $S = \{v_n : v_n < v_{n+1}\}$, where $S$ has no finite limit point, such that

$$\sigma''(v) = 0, \qquad \text{if } v \in S$$

and

$$\sigma''(v) < 0, \text{if } v \in (-\infty, v_1), \qquad \sigma''(v) > 0, \text{if } v \in (v_1, +\infty) - S,$$

($A_3$) for every $v_n \in S$, there exists a positive integer number $p_n$ such that

$$\sigma^{(2)}(v_n) = \sigma^{(3)}(v_n) = \cdots = \sigma^{(p_n - 1)}(v_n) = 0, \qquad \sigma^{(p_n)}(v_n) \neq 0.$$

*Remark* 1.1. When $\sigma(v)$ is an analytic function, assumption ($A_3$) holds automatically.

*Remark* 1.2. There exist many functions which satisfy the assumptions ($A_1$)–($A_3$). For example,

$$\sigma(v) = \int_0^v ds \int_0^s (\tau - v_1)^{2k_1 + 1} (\tau - v_2)^{2k_2} \cdots (\tau - v_n)^{2k_n} \, d\tau,$$

where $k_1, k_2, \ldots, k_n \geq 0$ are integer numbers, $v_1, v_2, \ldots, v_n \in \mathbf{R}$.

By simple calculation, the two eigenvalues of the system (1.1) are

$$\lambda_1 = -(\sigma'(v))^{1/2}, \qquad \lambda_2 = (\sigma'(v))^{1/2}, \tag{1.3}$$

with corresponding right eigenvectors

$$r_1 = \left(1, \sqrt{\sigma'(v)}\right)^T, \qquad r^2 = \left(1, -\sqrt{\sigma'(v)}\right)^T.$$

Riemann invariants are

$$z = u - \int_{v_1}^{v} \sqrt{\sigma'(s)} \, ds, \qquad w = u + \int_{v_1}^{v} \sqrt{\sigma'(s)} \, ds. \tag{1.4}$$

By direct calculation, we have

$$\nabla\lambda_1 \cdot r_1 = -\frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}, \qquad \nabla\lambda_2 \cdot r_2 = \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}. \tag{1.5}$$

Hence, the system (1.1) is strictly hyperbolic, and two characteristic fields are linearly degenerate only on the set $S$.

For the system (1.1), there have been many results on the global weak solutions. DiPerna [9] firstly gave the existence of the global weak solutions for the case when $v\sigma''(v) > 0$, $\forall v \in \mathbf{R} - \{0\}$. This is the first successful application of the theory of compensated compactness to a system of conservation laws. Based on DiPerna's fundamental result (see [9]), Lin [16] obtained the existence of the global weak solutions for the case when $v\sigma''(v) < 0$, $\forall v \in \mathbf{R} - \{0\}$ and initial data in $L^p$ space. Recently, Lu [18] extended (1.1) to a more general system

$$v_t + (u + g(v))_x = 0, \qquad u_t + (cu + f(v))_x = 0 \tag{1.6}$$

and also obtained the existence of the global weak solutions using DiPerna's framework in [9]. We observe, however, that all the above papers require that both characteristic fields for the system (1.1) be linearly degenerate only on the line $v = 0$, i.e., set $S$ is a single point one. In this paper, we are going to study the case when more linearly degenerate points appear for the system (1.1); that is, under the assumptions $(A_2)$, $(A_3)$, we will prove the existence of the global weak solution for the Cauchy problem (1.1), (1.2) by the method of compensated compactness. It is well known that linear degeneracy makes it difficult to reduce the family of Young measures uniquely determined by the viscosity solution sequence for the system (1.1) to a family of Dirac measures.

To overcome the difficulty caused by the linear degeneracy, we will construct the special entropy-entropy flux pairs of Lax type, which are different from the entropy pairs introduced by DiPerna [9], where all terms of progression are the functions of a single variable. Furthermore, we obtain the necessary uniform estimates on the major terms by applying the theory of singular perturbation of the ordinary differential equations. These special entropy pairs provide that the family of Young measures $\nu_{x,t}$ uniquely determined by the viscosity solutions for the system (1.1) is a family of Dirac masses. Therefore, the following convergence theorem on the viscosity solutions can be established by applying the theory of compensated compactness.

THEOREM 1.1 (Main theorem). *Let the initial data* $(v_0(x), u_0(x))$ *be bounded and measurable. Furthermore, assume that* $\sigma(v)$ *satisfies* $(A_1)$–$(A_3)$. *Then there exists a subsequence (still labeled)* $(v^\varepsilon(x,t), u^\varepsilon(x,t))$ *of the viscosity solutions given by* (2.1), (1.2) *and a bounded measurable function* $(v(x,t), u(x,t))$ *such that*

$$v^\varepsilon(x,t) \to v(x,t),\, u^\varepsilon(x,t) \to u(x,t), \qquad \text{a.e. on } \Omega,$$

*where* $\Omega \subset \mathbf{R} \times \mathbf{R}^+$ *is any bounded open set. Therefore,* $(v(x,t), u(x,t))$ *is an admissible solution of the Cauchy problem* (1.1), (1.2).

The plan of this paper is as follows: In Section 2, we study the viscosity solutions for the system (1.1). In Section 3, we construct four families of entropy-entropy flux of Lax type, which are used in Section 4 to prove that the family of Young measures $\nu_{x,t}$ is indeed a family of Dirac ones.

## 2. VISCOSITY SOLUTIONS

In this section we consider the Cauchy problem for the related parabolic system of (1.1)

$$\begin{cases} v_t - u_x = \varepsilon v_{xx}, \\ u_t - \sigma_x(v) = \varepsilon u_{xx}, \end{cases} \tag{2.1}$$

with initial data (1.2).

Along the lines of $w = N$, $w = -N$, $z = N$, and $z = -N$, we have from the assumptions $(A_1)$, $(A_2)$

$$w = N: \frac{du}{dv} = -(\sigma'(v))^{1/2} < 0,$$

$$\frac{d^2u}{dv^2} = -2(\sigma'(v))^{-1/2}\sigma''(v) \leq 0, \text{ for } v > v_1;$$

$$w = -N: \frac{du}{dv} = -(\sigma'(v))^{1/2} < 0,$$

$$\frac{d^2u}{dv^2} = -2(\sigma'(v))^{-1/2}\sigma''(v) > 0, \text{ for } v < v_1;$$

$$z = N: \frac{du}{dv} = (\sigma'(v))^{1/2} > 0,$$

$$\frac{d^2u}{dv^2} = 2(\sigma'(v))^{-1/2}\sigma''(v) \geq 0, \text{ for } v > v_1;$$

$$z = -N: \frac{du}{dv} = (\sigma'(v))^{1/2} > 0,$$

$$\frac{d^2u}{dv^2} = 2(\sigma'(v))^{-1/2}\sigma''(v) < 0, \text{ for } v < v_1.$$

Therefore, by applying the theory in [5] for general invariant regions, we have the following theorem.

THEOREM 2.1. *Let the assumptions* $(A_1)$, $(A_2)$ *hold. If the initial data* $v_0(x)$, $u_0(x)$ *are bounded and measurable, then*

$$\Sigma = \{(v,u): |w(v,u)| \le N, |z(v,u)| \le N\}$$

*is the invariant regions of* (2.1) *for all* $\varepsilon > 0$, *where N is a positive constant depending only on the initial data and* $\sigma$.

From Theorem 2.1, the solutions of the Cauchy problem (2.1), (1.2) have a $L^\infty$*-priori* estimate

$$|v^\varepsilon(x,t)| \le M, |u^\varepsilon(x,t)| \le M, \tag{2.2}$$

where $M$ is a positive constant, independent of $\varepsilon$. Therefore, the following global existence of solutions is obtained.

THEOREM 2.2. *Under the assumptions* $(A_1)$, $(A_2)$, *if the initial data* $v_0(x)$, $u_0(x)$ *are bounded measurable, then for any fixed* $\varepsilon > 0$, *the Cauchy problem* (2.1), (1.2) *admits a unique global smooth solution* $(v^\varepsilon(x,t), u^\varepsilon(x,t))$ *satisfying the estimates* (2.2).

Noticing that the system (1.1) admits a strictly convex entropy

$$\eta^* = \tfrac{1}{2}u^2 + \int_{v_1}^{v} \sigma(s)\,ds,$$

we have the following theorem (cf. [8, 25]).

THEOREM 2.3. *For any* $C^2$*-entropy pairs* $(\eta(v,u), q(v,u))$ *of the system* (1.1),

$$\eta(v^\varepsilon, u^\varepsilon)_t + q(v^\varepsilon, u^\varepsilon)_x$$

*is compact in* $H^{-1}_{loc}(\mathbf{R} \times \mathbf{R}^+)$.

Theorem 2.3 guarantees that that Tartar−Murat functional equation (3.2) in Section 4 is true. Equation (3.2) plays a very important role in reducing the family of Young measures $\nu_{x,t}$ which are uniquely determined by the viscosity solution sequence for the system (1.1) to a family of Dirac ones in Section 4.

## 3. ENTROPY WAVES

In this section, we study the entropy waves for the system (1.1). We will construct the Lax entropies and give the required estimates by using the theory of singular perturbation of the ordinary differential equations.

We recall that a pair of real-valued mappings $(\eta, q)$ is an entropy-entropy flux pair of (1.1) if for all smooth solutions of the system (1.1), we have

$$q_v = -\sigma'(v)\eta_u, \qquad q_u = -\eta_v. \tag{3.1}$$

Eliminating the $q$ from (3.1) yields

$$\eta_{vv} = \sigma'(v)\eta_{uu}. \tag{3.2}$$

For any $N \geq 1$, substituting entropies $\eta_k^1 = e^{kw}(\sum_{n=0}^N (a_n(v)/k^n) + (a_{N+1}(v,k)/k^{N+1}))$ into (3.2), we obtain that

$$\begin{aligned} &k\left(2\sqrt{\sigma'(v)}\,a_0' + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,a_0\right) \\ &\quad + \sum_{n=1}^N \frac{1}{k^{n-1}}\left(2\sqrt{\sigma'(v)}\,a_n' + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,a_n + a_{n-1}''\right) \\ &\quad + \frac{1}{k^N}\left(2\sqrt{\sigma'(v)}\,a_{N+1}' + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,a_{N+1} + a_N'' + \frac{a_{N+1}''}{k}\right) = 0. \end{aligned}$$

Let

$$2\sqrt{\sigma'(v)}\,a_0' + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,a_0 = 0 \tag{3.3}$$

and

$$2\sqrt{\sigma'(v)}\,a_n' + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,a_n + a_{n-1}'' = 0, \qquad 1 \leq n \leq N; \tag{3.4}$$

then one gets

$$2\sqrt{\sigma'(v)}\,a_{N+1}' + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,a_{N+1} + a_N'' + \frac{a_{N+1}''}{k} = 0. \tag{3.5}$$

Solving (3.3), (3.4) recursively with respect to $a_n$ $(n \geq 0)$, we get

$$a_0(v) = (\sigma'(v))^{-1/4} > 0, \tag{3.6}$$

$$a_n(v) = -\tfrac{1}{2}(\sigma'(v))^{-1/4}\int_{v_1}^{v} a_{n-1}''(s)(\sigma'(s))^{-1/4}\,ds, \qquad 1 \leq n \leq N. \tag{3.7}$$

Through tedious but simple calculation, we have by the method of induction

$$a_n'(v) = \frac{(-1)^{n+1}}{2^{n+2}} (\sigma'(v))^{-(1/4)(2n+5)} \sigma^{(n+2)}(v)$$
$$+ g_{n1}(v)\sigma^{(n+1)}(v) + \cdots + g_{nn}(v)\sigma^{(2)}(v), \qquad 1 \le n \le N, \quad (3.8)$$

where $g_{ni}(v) \in C^1(\mathbf{R})$, $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots, N$.

In order to get the existence of $a_{N+1}(v, k)$ and its uniform boundedness with respect to $k$, we first establish the following theorem:

THEOREM 3.1. *For sufficiently large $k$, Eq.* (3.5) *has $C^3$-solution $a_{N+1}(v, k)$ and $a_{N+1}(v, k)$, $a'_{N+1}(v, k)$ are uniformly bounded with respect to $k$.*

Before proving Theorem 3.1, we introduce the following theorem of singular perturbation of the ordinary differential equations (cf. [12]).

THEOREM 3.2. *Let $Y(x) \in C^2[\alpha, \beta]$ be the solution of the equation*

$$F(x, Y, Y') = 0,$$

*and functions $f(x, y, z, \lambda)$, $F(x, y, z)$ be continuous on the regions $\alpha \le x \le \beta$, $|y - Y(x)| \le P(x)$, $|z - Y'(x)| \le Q(x)$ for some positive functions $P(x)$, $Q(x)$ and $\lambda_0 > \lambda > 0$. In addition,*

$$\begin{cases} |f(x, y, z, \lambda) - F(x, y, z)| \le \delta, \\ |F(x, y_2, z) - F(x, y_1, z)| \le A\,|y_2 - y_1|, \\ \dfrac{F(x, y, z_2) - F(x, y, z_1)}{z_2 - z_1} \ge L \end{cases} \tag{3.9}$$

*for some positive constants $\delta$, $A$, $L$.*

*If $y(x) = y(x, \lambda)$ is a solution of the following second order ordinary differential equation*

$$\lambda y'' + f(x, y, y', \lambda) = 0,$$

*with $y(x_0) = Y(x_0)$ and $y'(x_0)$ being arbitrary, where $x_0 \in [\alpha, \beta]$, then for sufficiently small $\lambda > 0$, $\delta > 0$, and $B = |y'(x_0) - Y'(x_0)|$, $y(x)$ exists for all $\alpha \le x \le \beta$ and satisfies*

$$|y(x, \lambda) - Y(x)| < \left\{ \frac{\delta}{A} + \lambda\left(\frac{B}{L} + \frac{D}{A}\right)\right\} \exp\left(\frac{Ax}{L}\right),$$

*where $D = \max_{\alpha \le x \le \beta} |Y''(x)|$.*

*Proof of Theorem* 3.1. We know that

$$Y_1(v) = -\tfrac{1}{2}(\sigma'(v))^{-1/4}\int_{v_1}^{v} a_N''(s)(\sigma'(s))^{-1/4}\,ds$$

is a $C^2$-solution of equation

$$2\sqrt{\sigma'(v)}\,Y'(v) + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,Y(v) + a_N''(v) = 0$$

on $-M \le v \le M$.

Choosing

$$\begin{aligned} F(v, y, y') &= f\left(v, y, y', \frac{1}{k}\right) \\ &= 2\sqrt{\sigma'(v)}\,y'(v) + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\,y(v) + a_N''(v), \end{aligned}$$

$$B_1 = 0,\ \lambda = \frac{1}{k},\ L_1 = 2\inf_{|v|\le M}\sqrt{\sigma'(v)},\ \delta_1 = 0,\ A_1 = \sup_{|v|\le M}\frac{|\sigma''(v)|}{2\sqrt{\sigma'(v)}},$$

then it is easy to verify (3.9). By using Theorem 3.2, Eq. (3.5) has a solution $a_{N+1}(v,k)$ on $-M \le v \le M$ and

$$|a_{N+1}(v,k) - Y_1(v)| < \frac{D_1}{kA_1}\exp\left(\frac{A_1 v}{L_1}\right),$$

where $D_1 = \sup_{|v|\le M}|Y_1''(v)|$. Thus

$$|a_{N+1}(v,k)| \le |Y_1(v)| + \frac{D_1}{A_1}\exp\left(\frac{A_1 v}{L_1}\right). \tag{3.10}$$

Next we estimate $a_{N+1}'(v,k)$. To this end, differentiating equation (3.5) with respect to $v$, we get

$$\begin{aligned} &2\sqrt{\sigma'(v)}\,a_{N+1}''(v,k) + \frac{3\sigma''(v)}{2\sqrt{\sigma'(v)}}\,a_{N+1}'(v,k) \\ &\quad + \left(\frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\right)' a_{N+1}(v,k) + a_N'''(v) + \frac{a_{N+1}'''(v,k)}{k} = 0. \end{aligned} \tag{3.11}$$

Let $Y_2(v)$ be a $C^2$-solution of equation

$$2\sqrt{\sigma'(v)}\,Y'(v) + \frac{3\sigma''(v)}{2\sqrt{\sigma'(v)}}\,Y(v) + \left(\frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\right)' Y_1(v) + a_N'''(v) = 0. \quad (3.12)$$

Then $Y_2(v)$ is uniformly bounded with respect to $k$.

Choosing

$$F(v, y, y') = 2\sqrt{\sigma'(v)}\,y' + \frac{3\sigma''(v)}{2\sqrt{\sigma'(v)}}y + \left(\frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\right)' Y_1(v) + a_N'''(v),$$

$$f\left(v, y, y', \frac{1}{k}\right) = 2\sqrt{\sigma'(v)}\,y' + \frac{3\sigma''(v)}{2\sqrt{\sigma'(v)}}y + \left(\frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\right)' a_{N+1}(v, k) + a_N'''(v),$$

$$\delta_2 = \frac{D_1}{kA_1} \sup_{|v|\le M} \left|\left(\frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\right)'\right| \exp\left(\frac{A_1 v}{L_1}\right), \qquad \lambda = \frac{1}{k},\ B_2 = 0,$$

$$A_2 = \sup_{|v|\le M} \frac{3|\sigma''(v)|}{2\sqrt{\sigma'(v)}}, \qquad L_2 = L_1,$$

then it is easy to verify (3.9) and it follows from Theorem 3.2

$$|a_{N+1}'(v, k) - Y_2(v)| \le \frac{D_2}{kA_2} \exp\left(\frac{A_2 v}{L_2}\right),$$

where $D_2 = \sup_{|v| \le M} |Y_2''(v)|$. Thus

$$|a_{N+1}'(v, k)| \le |Y_2(v)| + \frac{D_2}{A_2} \exp\left(\frac{A_2 v}{L_2}\right). \quad (3.13)$$

The proof of Theorem 3.1 is completed. ∎

Thus using (3.1) and the argument above, a progressing wave of the system (1.1) is provided by

$$\begin{cases} \eta_k^1 = e^{kw}\left( \sum_{n=0}^{N} \dfrac{a_n(v)}{k^n} + \dfrac{a_{N+1}(v,k)}{k^{N+1}} \right), \\ q_k^1 = \lambda_1 \eta_k^1 - \dfrac{1}{k} e^{kw}\left( \sum_{n=0}^{N} \dfrac{a_n'(v)}{k^n} + \dfrac{a_{N+1}'(v,k)}{k^{N+1}} \right), \end{cases} \tag{3.14}$$

where $a_{N+1}(v,k)$, $a'_{N+1}(v,k)$ exist, and are uniformly bounded with respect to $k$.

Similarly, we can obtain other entropy-entropy flux pairs of Lax type as

$$\begin{cases} \eta_{-k}^1 = e^{-kw}\left( \sum_{n=0}^{N} \dfrac{b_n(v)}{k^n} + \dfrac{b_{N+1}(v,k)}{k^{N+1}} \right), \\ q_{-k}^1 = \lambda_1 \eta_{-k}^1 + \dfrac{1}{k} e^{-kw}\left( \sum_{n=0}^{N} \dfrac{b_n'(v)}{k^n} + \dfrac{b_{N+1}'(v,k)}{k^{N+1}} \right), \end{cases} \tag{3.15}$$

$$\begin{cases} \eta_k^2 = e^{kz}\left( \sum_{n=0}^{N} \dfrac{c_n(v)}{k^n} + \dfrac{c_{N+1}(v,k)}{k^{N+1}} \right), \\ q_k^2 = \lambda_2 \eta_k^2 - \dfrac{1}{k} e^{kz}\left( \sum_{n=0}^{N} \dfrac{c_n'(v)}{k^n} + \dfrac{c_{N+1}'(v,k)}{k^{N+1}} \right) \end{cases} \tag{3.16}$$

and

$$\begin{cases} \eta_{-k}^2 = e^{-kz}\left( \sum_{n=0}^{N} \dfrac{d_n(v)}{k^n} + \dfrac{d_{N+1}(v,k)}{k^{N+1}} \right), \\ q_{-k}^2 = \lambda_2 \eta_{-k}^2 + \dfrac{1}{k} e^{-kz}\left( \sum_{n=0}^{N} \dfrac{d_n'(v)}{k^n} + \dfrac{d_{N+1}'(v,k)}{k^{N+1}} \right), \end{cases} \tag{3.17}$$

where $b_n(v)$, $c_n(v)$, $d_n(v)$ $(n = 1, 2, \dots, N)$ and $b_{N+1}(v,k)$, $c_{N+1}(v,k)$, $d_{N+1}(v,k)$ satisfy

$$b_0(v) = a_0(v) > 0, \tag{3.18}$$

$$2\sqrt{\sigma'(v)}\, b_n' + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\, b_n - b_{n-1}'' = 0, \qquad 1 \le n \le N, \tag{3.19}$$

$$2\sqrt{\sigma'(v)}\, b'_{N+1} + \frac{\sigma''(v)}{2\sqrt{\sigma'(v)}}\, b_{N+1} - b''_N - \frac{b''_{N+1}}{k} = 0, \tag{3.20}$$

$$c_n(v) = b_n(v),\, 0 \le n \le N, \qquad c_{N+1}(v,k) = b_{N+1}(v,k), \tag{3.21}$$

$$d_n(v) = a_n(v),\, 0 \le n \le N, \qquad d_{N+1}(v,k) = a_{N+1}(v,k) \tag{3.22}$$

and

$$b_n(v) = \tfrac{1}{2}(\sigma'(v))^{-1/4} \int_{v_1}^{v} b''_{n-1}(s)(\sigma'(s))^{-1/4}\, ds, \qquad 1 \le n \le N,$$

$$\begin{aligned} b'_n(n) = {} & \frac{-1}{2^{n+2}}(\sigma'(v))^{-(1/4)(2n+5)} \sigma^{(n+2)}(v) \\ & + h_{n1}(v)\sigma^{(n+1)}(v) + \cdots + h_{nn}(v)\sigma^{(2)}(v), \qquad 1 \le n \le N, \end{aligned} \tag{3.23}$$

where $h_{ni}(v) \in C^1(\mathbf{R})$, $i = 1, 2, \ldots, n$, $n = 1, 2, \ldots, N$.

In addition, we can prove the existence of $b_{N+1}(v,k)$ and the uniform boundedness of $b_{N+1}(v,k)$, $b'_{N+1}(v,k)$ with respect to $k$.

Furthermore, we have

$$\begin{cases} \eta_k^1 = e^{kw}\left(a_0(v) + O\left(\dfrac{1}{k}\right)\right), \qquad \eta_{-k}^1 = e^{-kw}\left(b_0(v) + O\left(\dfrac{1}{k}\right)\right), \\ \lambda_1 \eta_k^1 - q_k^1 = \dfrac{1}{k} e^{kw}\left(\displaystyle\sum_{n=0}^{N} \frac{a'_n(v)}{k^n} + O\left(\frac{1}{k^{N+1}}\right)\right) \end{cases} \tag{3.24}$$

and

$$\begin{cases} \eta_k^2 = e^{kz}\left(c_0(v) + O\left(\dfrac{1}{k}\right)\right), \qquad \eta_{-k}^2 = e^{-kz}\left(d_0(v) + O\left(\dfrac{1}{k}\right)\right), \\ q_{-k}^2 - \lambda_2 \eta_{-k}^2 = \dfrac{1}{k} e^{-kz}\left(\displaystyle\sum_{n=0}^{N} \frac{d'_n(v)}{k^n} + O\left(\frac{1}{k^{N+1}}\right)\right). \end{cases} \tag{3.25}$$

The properties (3.24), (3.25) are basic to our analysis in the next section.

## 4. PROOF OF THEOREM 1.1

Before proving Theorem 1.1, we first state the following lemmas.

LEMMA 4.1. *Let $K \subset \mathbf{R}^n$ be a bounded open set and $U^\varepsilon : \mathbf{R} \times \mathbf{R}^+ \to \mathbf{R}^n$*

*be a sequence of measurable functions such that*

$$U^{\varepsilon}(x,t) \in K, \qquad a.e.$$

*and, for function pairs* $(\eta_i, q_i)$ $(i = 1,2)$,

$$\eta_i(U^{\varepsilon})_t + q_i(U^{\varepsilon})_x \text{ compact in } H^{-1}_{loc}.$$

*Then*

(i) *There exists a subsequence (still labeled)* $U^{\varepsilon}$ *and a family of Young measures*

$$\nu_{x,t} \in \mathrm{Prob}(\mathbf{R}^n), \qquad \mathrm{supp}\,\nu_{x,t} \subset \overline{K},$$

*such that*

$$\mathrm{w}^* - \lim g(U^{\varepsilon}) = \langle \nu_{x,t}(\lambda), g(\lambda)\rangle = \int_{\mathbf{R}^n} g(\lambda)\, d\nu_{x,t}(\lambda), \tag{4.1}$$

*for any continuous function* $g$, *and the Tartar–Murat functional equation*

$$\langle \nu_{x,t}, \eta_1 q_2 - \eta_2 q_1\rangle = \langle \nu_{x,t}, \eta_1\rangle\langle \nu_{x,t}, q_2\rangle - \langle \nu_{x,t}, \eta_2\rangle\langle \nu_{x,t}, q_1\rangle, \qquad a.e. \tag{4.2}$$

*holds for the function pairs* $(\eta_i, q_i)$ $(i = 1,2)$, *where* w*-lim *denotes the weak limit in the weak-star topology.*

(ii) *The sequence* $U^{\varepsilon}$ *converges strongly to* $U$ *if and only if the family of Young measures* $\nu_{x,t}$ *at almost all points* $(x,t)$ *is a family of Dirac measures concentrated at* $U(x,t)$; *that is,* $\nu_{x,t} = \delta_{U(x,t)}$.

The detailed proof of this lemma can be found in [25].

*Remark* 4.1. The family of Young measures $\nu_{x,t} \in \mathrm{Prob}(\mathbf{R}^n)$ is completely determined by the sequence $U^{\varepsilon}(x,t)$.

LEMMA 4.2. *Let the assumption* $(A_1)$–$(A_3)$ *be satisfied. Then for any* $n$

$$\sigma^{(p_n)}(v_n) > 0,$$

*and* $p_1 \geq 3$ *is an odd number,* $p_n \geq 4$ $(n \geq 2)$ *is an even number.*

*Proof.* When $n = 1$, from $\sigma^{(p_1)}(v_1) \neq 0$, there exists $\delta_1 > 0$ such that $(v_1 - \delta_1, v_1 + \delta_1) \subset (-\infty, v_2)$ and

$$\mathrm{sign}\big(\sigma^{(p_1)}(v)\big) = \mathrm{sign}\big(\sigma^{(p_1)}(v_1)\big) \qquad \text{for } v \in (v_1 - \delta_1, v_1 + \delta_1). \tag{4.3}$$

By applying Taylor's formula in $(v_1 - \delta_1, v_1 + \delta_1)$, we have

$$\sigma''(v) = \sigma''(v_1) + \sigma'''(v_1)(v - v_1) + \cdots + \frac{1}{(p_1 - 3)!}\sigma^{(p_1-1)}(v_1)(v - v_1)^{p_1-3} + \frac{1}{(p_1 - 2)!}\sigma^{(p_1)}(\theta v_1 + (1-\theta)v)(v - v_1)^{p_1-2}, \quad (4.4)$$

where $\theta \in (0,1)$.

From the assumptions $(A_2)$, $(A_3)$, the formula (4.4) deduces

$$\sigma^{(p_1)}(\theta v_1 + (1-\theta)v)(v - v_1)^{p_1-2} > 0, \qquad \text{if } v \in (v_1, v_1 + \delta_1) \quad (4.5)$$

and

$$\sigma^{(p_1)}(\theta v_1 + (1-\theta)v)(v - v_1)^{p_1-2} < 0, \qquad \text{if } v \in (v_1 - \delta_1, v_1), \quad (4.6)$$

and (4.3), (4.5) show

$$\sigma^{(p_1)}(v_1) > 0. \quad (4.7)$$

From (4.3), (4.6), and (4.7), one gets that $p_1 \geq 3$ is an odd number.

When $n \geq 2$, the proof is similar and the details are omitted. This completes the proof of Lemma 4.2. ∎

*Proof of Theorem* 1.1. From Theorem 2.2, Theorem 2.3, and Lemma 4.1, we conclude that there exists a family of probability measures $\{\nu_{x,t}\}$, supp $\nu_{x,t} \in \overline{K} = \{(v,u): |v(x,t)| \leq M, |u(x,t)| \leq M\}$, uniquely determined by the viscosity solutions $(v^\varepsilon(x,t), u^\varepsilon(x,t))$, such that the Tartar–Murat functional equation (4.2) holds for all $C^2$-entropy pairs $(\eta_i, q_i)$ $(i = 1,2)$ of the system (1.1).

Now we prove that the family of Young measures $\nu_{x,t}$ is a family of Dirac measures.

Let $Q$ denote the smallest characteristic rectangle

$$Q = \{(v,u): w^- \leq w(v,u) \leq w^+, z^- \leq z(v,u) \leq z^+\}$$

which contains the support of $\nu_{x,t}$.

As in [9], we introduce probability measures $\mu^i_{\pm k}$ $(i = 1,2)$ on $Q$ defined by

$$\langle \mu^i_{\pm k}, h\rangle = \langle \nu_{x,t}, h\eta^i_{\pm k}\rangle / \langle \nu_{x,t}, \eta^i_{\pm k}\rangle, \qquad i = 1,2, \qquad (4.8)_i$$

where $h = h(v,u)$ denotes an arbitrary continuous function. As a consequence of weak-star compactness, there exist probability measures $\mu^i_\pm$ on

$Q$ such that

$$\left\langle \mu^i_{\pm}, h \right\rangle = \lim_{k \to \infty} \left\langle \mu^i_{\pm k}, h \right\rangle, \tag{4.9$_i$}$$

after the selection of an appropriate subsequence. We observe that the measures $\mu^i_+$ and $\mu^i_-$ are respectively concentrated on the boundary sections of $Q$ associated with $w$, $z$, i.e.,

$$\operatorname{supp} \mu^1_{\pm} \subset I_w^{\pm} = \left\{ (v,u) : w = w^{\pm}, z^- \le z \le z^+ \right\} \tag{4.10}$$

and

$$\operatorname{supp} \mu^2_{\pm} \subset I_z^{\pm} = \left\{ (v,u) : w^- \le w \le w^+, z = z^{\pm} \right\}. \tag{4.11}$$

Noticing (3.24), (3.25), we can deduce by the method in [9] that

$$\left\langle \mu^i_+, \lambda_i \eta - q \right\rangle = \left\langle \mu^i_-, \lambda_i \eta - q \right\rangle, \qquad i = 1, 2 \tag{4.12$_i$}$$

for any $C^2$-entropy pair $(\eta, q)$.

From assumption $(\mathrm{A}_2)$ and (1.4), the two eigenvalues are genuinely nonlinear when restricted to the set

$$\{(v,u) : v < 0\} \qquad \text{or equivalently} \qquad \{(v,u) : w < z\}. \tag{4.13}$$

To show $\nu_{x,t}$ is a family of Dirac measures, it is sufficient to prove $w^+ = w^-$ and $z^+ = z^-$. Otherwise, one of the following holds:

($\alpha$) $w^+ > w^-$, $z^+ > z^-$;

($\beta$) $w^+ > w^-$, $z^+ = z^-$;

($\gamma$) $w^+ = w^-$, $z^+ > z^-$.

In case ($\alpha$), we have

$$I_w^+ = \left\{ (v,u) : w = w^+, z^- \le z \le z^+, z^+ \le w^+ \right\}, \tag{4.14}$$

$$I_z^- = \left\{ (v,u) : z = z^-, w^- \le w \le w^+, w^- \ge z^- \right\}. \tag{4.15}$$

(See Fig. 4.1).

Otherwise, if (4.14) is violated, that is, $z^+ > w^+$ (see Fig. 4.2), then from (4.6), the second eigenvalue field is genuinely nonlinear on $I_z^+$ and $\sigma''(v) < 0$ on $I_z^+$.

By (3.16), we have

$$\begin{aligned} \lambda_2 \eta_k^2 - q_k^2 &= \frac{1}{k} e^{kz} \left( c_0'(v) + O\left(\frac{1}{k}\right) \right) \\ &= -\frac{1}{4k} e^{kz} \left( (\sigma'(v))^{-5/4} \sigma''(v) + O\left(\frac{1}{k}\right) \right). \end{aligned} \tag{4.16}$$

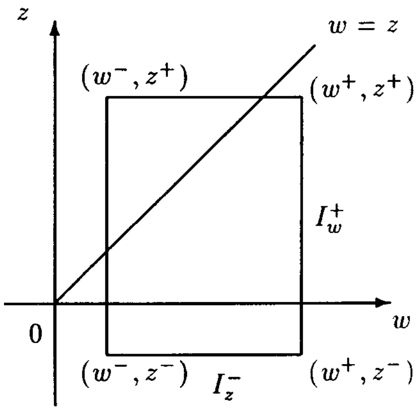


FIG. 4.1.

Thus there exist positive constants $c_1, c_2$ such that for sufficiently large $k$

$$\left\langle \mu_+^2 , \lambda_2 \eta_k^2 - q_k^2 \right\rangle \geq \frac{c_1}{k} e^{kz^+} \tag{4.17}$$

and

$$\left\langle \mu_-^2 , \lambda_2 \eta_k^2 - q_k^2 \right\rangle \leq \frac{c_2}{k} e^{kz^-}. \tag{4.18}$$

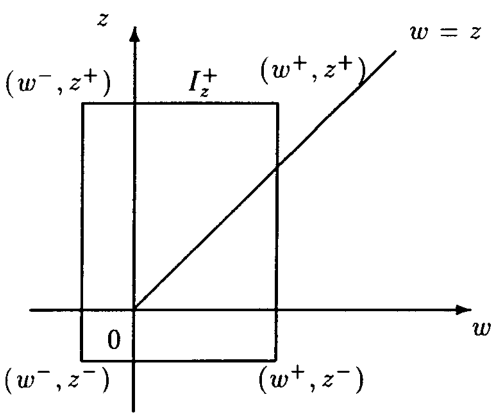


FIG. 4.2.

Combining (4.17), (4.18) with $(4.12)_2$, it is easy to get

$$\frac{c_1}{k}\, e^{kz^+} \le \frac{c_2}{k}\, e^{kz^-}. \tag{4.19}$$

Letting $k \to +\infty$ in (4.19), we conclude

$$z^+ = z^-.$$

This contradiction yields (4.14). Similarly, we can prove (4.15).

Now we will show case $(\alpha)$ is impossible by using (4.14) and (4.15). From assumption $(A_2)$ and (4.14), we know the first eigenvalue field is genuinely nonlinear on $I_w^+ - S'$, where

$$S' = \left\{ (w^+, z_m^+) : w^+ - z_m^+ = 2\int_{v_1}^{v_m} \sqrt{\sigma'(s)}\, ds, \forall v_m \in S \right\}. \tag{4.20}$$

Furthermore, it is easy to prove

$$\sigma''(v) \ge 0 \qquad \text{on } I_w^+ \tag{4.21}$$

and

$$\sigma''(v) > 0 \qquad \text{on } I_w^+ - S'. \tag{4.22}$$

Now we will show that

$$\phi \ne \operatorname{supp} \mu_+^1 \subset S'. \tag{4.23}$$

Otherwise, we can assume that there exists a point $(w^+, z_0) \in I_w^+ - S'$ such that

$$(w^+, z_0) \in \operatorname{supp} \mu_+^1 . \tag{4.24}$$

From (3.6), (3.14), we have

$$\begin{aligned} \lambda_1 \eta_k^1 - q_k^1 &= \frac{1}{k}\, e^{kw}\left( a_0'(v) + O\left(\frac{1}{k}\right)\right) \\ &= -\frac{1}{4k}\, e^{kw}\left( (\sigma'(v))^{-5/4} \sigma''(v) + O\left(\frac{1}{k}\right)\right). \end{aligned} \tag{4.25}$$

Combining (4.10), (4.21), (4.22), (4.24) with (4.25), there exists a positive constant $c_3$, independent of $k$, such that

$$\left|\left\langle \mu_+^1 , \lambda_1 \eta_k^1 - q_k^1 \right\rangle\right| \ge \frac{c_3}{4k}\, e^{kw^+} \tag{4.26}$$

for sufficiently large $k$.

On the other hand, for sufficiently large $k$, we have from (4.25)

$$\left|\left\langle \mu_-^1, \lambda_1\eta_k^1 - q_k^1\right\rangle\right| \le \frac{c_4}{4k} e^{kw^-}, \tag{4.27}$$

where $c_4$ is a positive constant independent of $k$.

Combining (4.26), (4.27) with $(4.12)_1$, we have

$$\frac{c_3}{4k} e^{kw^+} \le \frac{c_4}{4k} e^{kw^-}. \tag{4.28}$$

Letting $k \to +\infty$ in (4.28), we deduce $w^+ = w^-$, which proves (4.23).

From (4.23), we know that, for any functions $f(w, z)$, there exist points $(w^+, z_{m_1}^+), (w^+, z_{m_2}^+), \dots, (w^+, z_{m_l}^+)$ and constants $\alpha_1, \alpha_2, \dots, \alpha_l$ with $w^+ - z_{m_i}^+ = 2\int_{v_1}^{v_{m_i}} \sqrt{\sigma'(s)}\, ds$, $v_{m_i} \in S$, $i = 1, 2, \dots, l$, and $\alpha_i > 0$, $i = 1, 2, \dots, l$, $\sum_{i=1}^{l} \alpha_i = 1$, such that

$$\left\langle \mu_+^1, f\right\rangle = \sum_{i=1}^{l} \alpha_i f\left(w^+, z_{m_i}^+\right). \tag{4.29}$$

Choosing $f(w, z) = \lambda_1\eta_k^1 - q_k^1$ in (4.29), we have from (3.14), (3.8)

$$\begin{aligned}
&\left\langle \mu_+^1, \lambda_1\eta_k^1 - q_k^1\right\rangle \\
&\quad = \frac{1}{k} e^{kw^+} \sum_{i=1}^{l} \alpha_i \left\{ \sum_{n=0}^{N} \frac{a_n'(v_{m_i})}{k^n} + O\left(\frac{1}{k^{N+1}}\right)\right\} \\
&\quad = \frac{1}{k} e^{kw^+} \sum_{i=1}^{l} \alpha_i \sum_{n=0}^{N} \frac{1}{k^n} \left( \frac{(-1)^{n+1}}{2^{n+2}} \left(\sigma'(v_{m_i})\right)^{-(1/4)(2n+5)} \sigma^{(n+2)}(v_{m_i})\right) \\
&\qquad + \frac{1}{k} e^{kw^+} \sum_{i=1}^{l} \alpha_i \sum_{n=0}^{N} \frac{1}{k^n} \Big( g_{n1}(v_{m_i}) \sigma^{(n+1)}(v_{m_i}) \\
&\qquad\qquad + \cdots + g_{nn}(v_{m_i}) \sigma^{(2)}(v_{m_i})\Big) \\
&\qquad + \frac{1}{k} e^{kw^+} O\left(\frac{1}{k^{N+1}}\right).
\end{aligned} \tag{4.30}$$

Let

$$p = \min(p_{m_1}, p_{m_2}, \dots, p_{m_l}), \qquad N + 2 = p;$$

then, from (4.30) and Lemma 4.2, there exists a positive constant $c_5$ such that for sufficiently large $k$

$$\left|\left\langle \mu_+^1, \lambda_1 \eta_k^1 - q_k^1 \right\rangle\right| \geq \frac{c_5}{2^p k^{p-1}} e^{kw^+}. \tag{4.31}$$

Combining (4.31), (4.27) with $(4.12)_1$, we have

$$\frac{c_5}{2^p k^{p-1}} e^{kw^+} \leq \frac{c_4}{4k} e^{kw^-}. \tag{4.32}$$

Letting $k \to +\infty$ in (4.32), we deduce $w^+ = w^-$, which shows case $(\alpha)$ is impossible.

For case $(\beta)$, we have

$$\operatorname{supp} \mu_-^1 = \{(w^-, z^-)\}, \qquad \operatorname{supp} \mu_+^1 = \{(w^+, z^+)\}.$$

Thus $(4.12)_1$ shows

$$\left(\lambda_1 \eta_k^1 - q_k^1\right)(w^+, z^+) = \left(\lambda_1 \eta_k^1 - q_k^1\right)(w^-, z^-),$$

i.e.,

$$\frac{1}{k} e^{kw^+} \left( \sum_{n=0}^{N} \frac{a_n'(v^+)}{k^n} + O\left(\frac{1}{k^{N+1}}\right)\right)$$
$$= \frac{1}{k} e^{kw^-} \left( \sum_{n=0}^{N} \frac{a_n'(v^-)}{k^n} + O\left(\frac{1}{k^{N+1}}\right)\right), \tag{4.33}$$

where $v^+$ and $v^-$ are defined by the implicit functions

$$w^+ - z^+ = 2\int_{v_1}^{v^+} \sqrt{\sigma'(s)}\, ds, \qquad w^- - z^- = 2\int_{v_1}^{v^-} \sqrt{\sigma'(s)}\, ds.$$

If $v^+ \overline{\in} S$, then by letting $k \to +\infty$ in (4.33), we deduce $w^+ = w^-$, which shows case $(\beta)$ is impossible.

If $v^+ \in S$, then we can deduce (4.32) similarly and get $w^+ = w^-$, which shows case $(\beta)$ is impossible also.

Similarly, case $(\gamma)$ is impossible also. Therefore, the argument above shows that the family of Young measures $\nu_{x,t}$ is a family of Dirac masses and the proof of Theorem 1.1 is completed by using Lemma 4.1. ∎

*Remark* 4.2. If we replace $(A_2)$ by the following assumption $(A_2')$:

$(A_2')$ there exists a finite or infinite set $S = \{v_n : v_n > v_{n+1}\}$, where $S$ has no finite limit point, such that

$$\sigma''(v) = 0, \qquad \text{if } v \in S$$

and

$$\sigma''(v) < 0, \text{ if } v \in (-\infty, v_1) - S, \qquad \sigma''(v) > 0, \text{ if } v \in (v_1, +\infty),$$

then the conclusions of Theorem 4.1 still hold.

*Remark* 4.3. In a similar way, we can obtain the existence of the global weak solution for the Cauchy problem (1.6), (1.2).

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